SOME PHYSICS OF A TWO-LAYER COATING UPON A GLASS SUBSTRATE

Mathematical outline

For a single layer j with refractive index n_j and phase thickness δ_j shown with normally incident light with (angular) repetency k_j, the characteristic (transfer) matrix is

$$M_j = \begin{bmatrix} \cos(\delta_j) & \frac{i}{n_j}\sin(\delta_j) \\ \frac{i}{n_j}\sin(\delta_j) & \cos(\delta_j) \end{bmatrix},$$

where i is the imaginary unit. [1] For the situation where one coats two layers of differing refractive indices upon a glass substrate that will be surrounded by air $(\text{set } n_0 = 1.00)$, and assuming the optical path lengths for both the inner and outer layer are equal $(\delta = k_I \cdot l_I = k_O \cdot l_O)$, the total characteristic matrix of the assembly, M_T is

$$M_T = M_I \cdot M_O = \begin{bmatrix} \cos(\delta) & \frac{i}{n_I}\sin(\delta) \\ \frac{i}{n_I}\sin(\delta) & \cos(\delta) \end{bmatrix} \cdot \begin{bmatrix} \cos(\delta) & \frac{i}{n_O}\sin(\delta) \\ \frac{i}{n_O}\sin(\delta) & \cos(\delta) \end{bmatrix} = \begin{bmatrix} \cos^2(\delta) - \frac{n_O}{n_I}\sin^2(\delta) & i\left(\frac{1}{n_I} + \frac{1}{n_O}\right)\sin(\delta)\cos(\delta) \\ i(n_I + n_O)\sin(\delta)\cos(\delta) & \cos^2(\delta) - \frac{n_I}{n_O}\sin^2(\delta) \end{bmatrix}.$$

The amplitude reflectivity r of an assembly is given by the relation

$$r = \frac{n_0 B - C}{n_0 B + C},$$

where $\begin{bmatrix} B \\ C \end{bmatrix}$ is the characteristic matrix of the assembly multiplied by the substrate vector $\begin{bmatrix} 1 \\ n_S \end{bmatrix}$. [2] Now, let the thicknesses of both layers be $l_j = \frac{\lambda_{j-1}}{4 n_j}$. The phase thickness due the light's propagation through one film is $\delta_j = k_{j-1} n_j l_j \cos(\vartheta_j) = \frac{2\pi}{\lambda_{j-1}} n_j l_j$ for normal incidence $(\text{ie } \vartheta = 0)$. So, setting $\delta = \frac{\pi}{2}$ for this case, we have

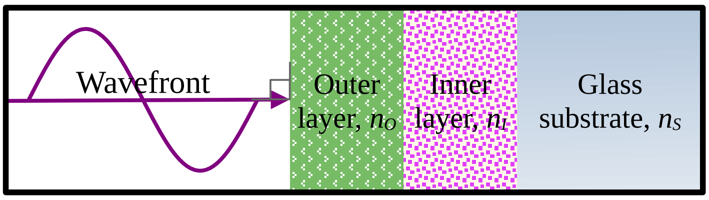
$$M_T\left(\delta = \frac{\pi}{2}\right) = \begin{bmatrix} 0 - \frac{n_O}{n_I}\cdot 1 & i\left(\frac{1}{n_I} + \frac{1}{n_O}\right)\cdot 1 \cdot 0 \\ i(n_I + n_O)\cdot 1 \cdot 0 & 0 - \frac{n_I}{n_O}\cdot 1 \end{bmatrix} = \begin{bmatrix} -\frac{n_O}{n_I} & 0 \\ 0 & -\frac{n_I}{n_O} \end{bmatrix},$$

and $\begin{bmatrix} B \\ C \end{bmatrix} = M_T \cdot \begin{bmatrix} 1 \\ n_S \end{bmatrix} \Rightarrow B = -\frac{n_I}{n_O} \text{ and } C = -\frac{n_O n_S}{n_I}$. Thus, the reflectivity of this assembly is

$$r = \frac{n_0 \cdot \left(-\frac{n_I}{n_O}\right) - \left(-\frac{n_O n_S}{n_I}\right)}{n_0 \cdot \left(-\frac{n_I}{n_O}\right) + \left(-\frac{n_O n_S}{n_I}\right)} \stackrel{\star}{=} \frac{n_0 n_I^2 - n_S n_O^2}{n_0 n_I^2 + n_S n_O^2},$$

where at the star equals $(\stackrel{\star}{=})$ the expression is multiplied by the following chosen one $\frac{-n_O n_I}{-n_O n_I}$. [3]

Numerical example

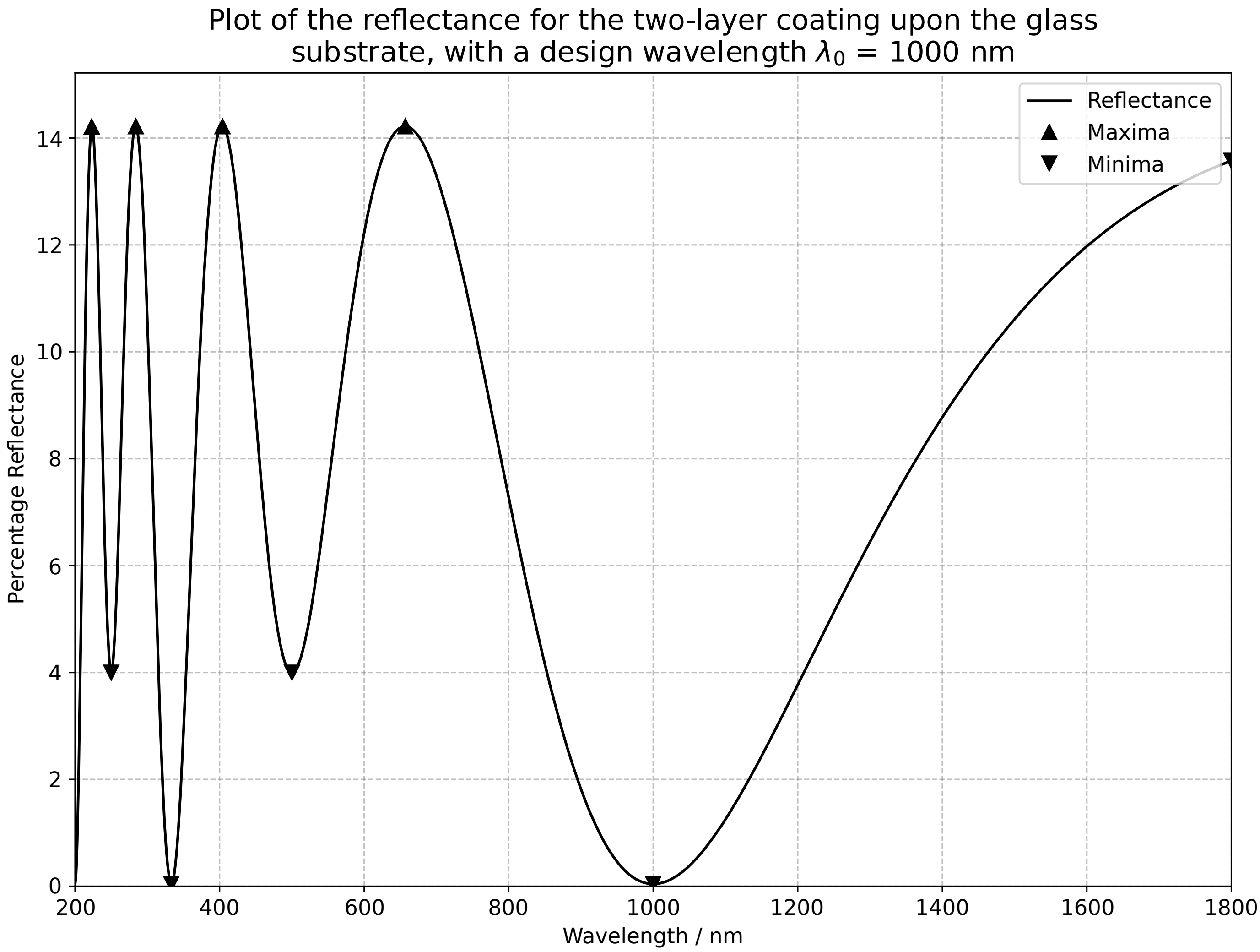


Set the refractive index of the glass substrate, $n_S = 1.5$. For maximum anti-reflectance (AR), the layer composition must have $n_O < n_I$ to give a phase shift of π^c at each interface.

The reflectance R is the square of the reflectivity. [4] If the outer layer and the inner layer have refractive indices of 1.6 and 2.0 respectively, $R \approx 0.0416\,\%$, indicating almost no reflection. Equivalently, there is almost perfect transmission with this double-layer coating. Ideally for complete AR $(R=0)$, with $n_O = 1.6$, then setting the numerator of the reflectivity to zero yields the bestest value for $n_I = \sqrt{3.84} \approx 1.96$.

Targeting for an AR coating for normally incident light at a (design) wavelength of 1000 nm, the variation in reflectance for light of different wavelengths is shown to fluctuate, with repetitive global minima and maxima. This was found with the Python code linked here.

Results



Featured Point	**Wavelength / nm**	**Reflectance, *R* in %**
Maximum	223	14.21
Maximum	284	14.22
Maximum	404	14.22
Maximum	657	14.22
Minimum	250	4.00
Minimum	333	0.04
Minimum	500	4.00
Minimum	1000	0.04
Boundary	1800	13.58

Notes

[1] See § 22 of Pedrotti³ (2017). *Introduction to Optics* (3rd ed.). Cambridge University Press. doi:10.1017/9781108552493.

[2] See §§ 2.5 and 2.7 of MacLeod, H A (2010). *Thin-Film Optical Filters* (4th ed.). CRC Press. doi:10.1201/9781420073034.

[3] The **chosen one** is a multiplicative factor that equals one but is written in a convenient form (normally a fraction) to make an expression less cumbersome. In this case, the chosen one used removes the ugly, foul, ever-so-stinky stacked fractions of sin. The name comes from my old A Level class, where I gather that they were quite fond of the *Revenge of the Sith*.

[4] See § 4.4 of Fowles, G R (1989). *Introduction to Modern Optics* (2nd, Dover ed.). Dover Publications. ISBN-13: 978-0-486-65957-2.

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